Indian Statistical Institute, Bangalore
B. Math. First Year, First Semester
Probability Theory: Mid-term Examination
Duration: 3 hours Maximum marks: 100 Date : 18-09-2007

1. Let $A_1, A_2, \ldots, A_n$ be n-events in a probability space. Show that
$$P(A_1 \bigcap A_2) \geq P(A_1) + P(A_2) - 1.$$
More generally, show that
$$P(A_1 \bigcap A_2 \bigcap \cdots \bigcap A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$
[15]

2. Let $N \geq 2$ be a natural number. Suppose an urn contains $(N-1)$ black balls and one green ball. Let X be the number of balls drawn until the green ball is drawn when sampling is done with replacement. Find the distribution and the expectation of X. Now suppose Y is the number of balls drawn until the green ball is drawn when sampling is done without replacement. Find the distribution and expectation of Y. [20]

3. Choose a natural number X at random from 1 to M. Let A be the event that 'X is a prime number' and let B be the event that 'X is at most 10'. Show that A and B are independent if $M = 20$ and they are not independent if $M = 21$. [10]

4. Suppose box I contains 2000 screws, 5 percent of which are defective, box II contains 500 screws, 40 percent of which are defective, box III contains 1000 screws, 10 percent of which are defective and box IV contains 1000 screws, 10 percent of which are defective. Now suppose you select a box at random and then pick up a screw at random what is the probability that it is defective? Given that the screw you chose is defective what is the probability that it came from box II? [15]

5. Suppose that you have a coin where the chance of H is p, with $0 \leq p \leq 1$ and you go on tossing until you get either TT or HH. Let R be the number of tosses required. Find the probability distribution of R. [20]

6. Let Z be a random variable with distribution function $F : R \to R$, given by
$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/3 & \text{if } -1 \leq x < 0 \\ 2/3 & \text{if } 0 \leq x < \frac{1}{2} \\ 4/5 & \text{if } \frac{1}{2} \leq x < 1 \\ 1 & \text{if } 1 < x \end{cases}$$
Find the probability mass function of Z. Suppose $W = 1 - 2Z$. Find the probability distribution function of W. [20]

7. Let $n \geq 3$ be a natural number and let S be the collection of all n-tuples $(d_1, d_2, \ldots, d_n)$, where each $d_i \in \{0, 1\}$ and $d_1 + d_2 + \ldots + d_n$ is an even number. Pick an element at random from S. Let A_k be the event that the k-th co-ordinate of the tuple you have picked is 1. Show that $A_1, A_2, \ldots, A_n$ are not independent but if you drop any of the A_i's the remaining are mutually independent. [10]